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## Generalized n-Potent Rings

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GENERALIZED  $n$ -POTENT RINGS

Dedicated to the memory of Professor Hisao Tominaga

HOWARD E. BELL, HAL G. MOORE, and ADIL YAQUB

In [1], [2], [3] and [4], some generalizations of Jacobson's theorem which states that a ring  $R$  in which for every  $a \in R$  there exists an integer  $n(a) > 1$ , depending on  $a$ , such that  $a^{n(a)} = a$ , is necessarily commutative have been studied in various directions. In this note, these results will be partially generalized to a wider class of rings, namely generalized  $n$ -potent rings.

Throughout,  $R$  denotes an associative ring,  $N$  the set of nilpotent elements of  $R$ ,  $C$  the center of  $R$ ,  $J$  the Jacobson radical of  $R$ ,  $C(R)$  the commutator ideal of  $R$ , and  $\mathbb{Z}$  the ring of rational integers. For  $x, y$  in  $R$ ,  $[x, y] = xy - yx$  denotes the commutator.

We now introduce the following definitions.

**Definition 1.** Let  $n$  be a fixed integer,  $n > 1$ . A ring  $R$  is called a *generalized  $n$ -potent ring* if

$$(*) \quad x^n - x \in N \cap C \quad \text{for all } x \in R \setminus (N \cup C).$$

**Definition 2.** If the set  $E$  of all idempotents of  $R$  is contained in  $C$  then  $R$  will be called a *ring with all idempotents central*. If  $N$  is contained in  $C$  then  $R$  will be called a *ring with all nilpotents central*.

Our main result is the following: *A generalized  $n$ -potent ring  $R$  with all idempotents central is commutative if it satisfies two conditions:*

- (i) For all  $a, b \in N$ ,  $[a, b] = [a, b]^q$  for some integer  $q > 1$ .
- (ii)  $(n - 1)[a, x] = 0$  implies  $[a, x] = 0$  for all  $a \in N$ ,  $x \in R$ .

It is further shown that all of the hypotheses of this theorem are essential.

We also prove a following structure theorem for generalized  $n$ -potent rings: If  $R$  is a generalized  $n$ -potent ring with all idempotents central which satisfies the above condition (ii), then  $R = C$  or  $R = N$ .

**Remark.** A ring  $R$  is a generalized  $n$ -potent ring if and only if one of the following conditions holds:

(1)  $R = N \cup C$ .

(2)  $R \neq N \cup C$  and  $(*)$  is satisfied.

Now, we take a closer look at case (1). In this case, for any  $a, b \in N$  with  $a - b \in C$ , we have  $ab = ba$ , and so  $a - b \in N$ . Moreover, for any  $a \in N$  and  $x \in R$  with  $ax \in C$ , an easy induction shows that  $(ax)^k = a^k x^k$  for all positive integers  $k$ , and so  $ax \in N$ . Similarly,  $xa \in N$ . Hence, if  $R = N \cup C$ , then  $N$  is an ideal of  $R$ , whence  $R = N$  or  $R = C$ . Next, for any elements  $a, b \in R$  with

$$[a, b] = [a, b]^q \quad \text{for some integer } q > 1$$

we have

$$[a, b] = ([a, b]^q)^q = [a, b]^{q^2} = [a, b]^{q^k}$$

for all positive integer  $k$ . Hence, if  $R = N$  and  $R$  satisfies (i) above, then  $R = N = C$ . Therefore, it follows that, if  $R = N \cup C$  and  $R$  satisfies (i) also, then  $R = C$ .

We have thus shown the following:

(a) If  $R = N \cup C$ , then  $R = N$  or  $R = C$ .

(b) If  $R = N \cup C$  and  $R$  satisfies (i) above, then  $R = C$ .

We now prove the following two lemmas.

**Lemma 1.** *If  $R$  is a generalized  $n$ -potent ring, then  $J \subseteq N \cup C$ .*

*Proof.* Suppose not, and let  $x \in J$ ,  $x \notin N$ ,  $x \notin C$ . Then, since  $R$  is generalized  $n$ -potent,  $x^n - x \in N \cap C$ . So  $(x^n - x)^m = 0$  for some  $m \in \mathbb{Z}^+$ , and thus

$$x^m = x^m(xg(x)), \quad \text{for some } g(\lambda) \text{ in } \mathbb{Z}[\lambda].$$

From this equation we readily obtain the relation  $x^m = x^m(xg(x))^m$ ; so we see that  $e = (xg(x))^m = x^m g(x)^m$  is idempotent. Thus,  $e = (xg(x))^m$  is an idempotent element of  $J$  (since  $x \in J$ ). Therefore  $e = 0$ . This implies that  $x^m = 0$ , which contradicts  $x \notin N$ . This proves the lemma.

**Lemma 2.** *Let  $R$  be a generalized  $n$ -potent ring with all idempotents central. Then,*

(i)  $ax \in N$  for all  $a \in N$  and  $x \in R$ ;

(ii)  $N$  is an ideal of  $R$ ;

(iii)  $C(R) \subseteq N \subseteq J \subseteq N \cup C$ .

*Proof.* Suppose that (i) is false and let  $a \in N$  and  $x \in R$ , with  $ax \notin N$ . As seen in above Remark, we note that if  $ax \in C$ , then  $ax \in N$ , a contradiction. This shows that  $ax \notin C$ . Thus,  $ax \notin N \cup C$ . So, because  $R$  is generalized  $n$ -potent, we have  $(ax)^n - ax \in N$ . Then, as in the proof of Lemma 1,  $(ax)^m = (ax)^m e$ , where  $e = ((ax)g(ax))^m$  is idempotent, and  $g(\lambda)$  in  $\mathbb{Z}[\lambda]$ .

By hypothesis,  $e$  is central so  $e = ee = e((ax)g(ax))^m = eat = aet$ , for some  $t \in R$ . Thus,  $e = aet$ , and as we noted in above Remark, since  $aet \in C$ ,  $e = e^q = (aet)^q = a^q(et)^q = a^q et^q$ , for all positive integers  $q$ . But since  $a \in N$ , for some  $q$ ,  $a^q = 0$ ; so  $e = 0$ . Hence,  $(ax)^m = 0$ , a contradiction, since  $ax \notin N$ . This proves part (i).

For (ii), let  $a \in N$  and  $x \in R$ . Then, from (i),  $ax \in N$ , and hence  $ax$  is right quasiregular for all  $x \in R$ . Thus  $a \in J$ , and hence  $N \subseteq J$ . Combining this with Lemma 1 we see that  $N \subseteq J \subseteq N \cup C$ . Now suppose that  $a, b \in N$ . Then both  $a$  and  $b \in J$ , so  $a - b \in J \subseteq N \cup C$ . Thus,  $a - b \in N$  or  $a - b \in C$ . But if  $a - b \in C$ , then  $ab = ba$  and, therefore,  $a - b \in N$ , for all  $a, b \in N$  in any case. Next suppose that  $a \in N$  and  $r \in R$ . Then, from part (i),  $ar \in N$ , say  $(ar)^q = 0$ . Hence,  $(ra)^{q+1} = 0$  and so  $ra \in N$ . Thus,  $N$  is an ideal of  $R$ .

Finally for (iii), since  $N$  is an ideal, the factor ring  $R/N$  exists. Since generalized  $n$ -potency is inherited by homomorphic images of  $R$ ,  $R/N$  is also generalized  $n$ -potent and we readily obtain that for every  $y \in R/N$ ,  $y^n - y$  is in the center of  $R/N$ . Therefore, by Herstein's Theorem [1],  $R/N$  is commutative, and thus  $C(R) \subseteq N$  [ $C(R)$  is the commutator ideal of  $R$ ]. Combining this result with  $N \subseteq J \subseteq N \cup C$ , we have part (iii).

We next prove the following theorems.

**Theorem 1.** *A generalized  $n$ -potent ring with all nilpotents central is commutative.*

*Proof.* Since the set  $N$  of nilpotent elements of the ring  $R$  is contained in the center  $C$  of  $R$ , this implies at once that  $x^n - x \in C$  for all  $x \in R$ . Hence, by Herstein's Theorem [1],  $R$  is commutative.

**Theorem 2.** *Suppose that  $R$  is a generalized  $n$ -potent ring with identity. Suppose further that for all  $a \in N$ ,  $x \in R$ ,*

$$(**) \quad (n-1)[a, x] = 0 \quad \text{implies} \quad [a, x] = 0.$$

Then  $R$  is commutative.

*Proof.* We claim that in this case  $N \subseteq C$ . Suppose, not, and let  $a \in N, a \notin C$ . Suppose that

$$(1) \quad [a^\sigma, x] = 0, \quad \text{for all } \sigma \geq \sigma_0, \sigma_0 \text{ minimal, } x \in R \text{ arbitrary.}$$

We claim that  $\sigma_0 = 1$ . Suppose not. Then,  $1 + a^{\sigma_0-1} \notin N \cup C$ , and hence,

$$(2) \quad (1 + a^{\sigma_0-1})^n - (1 + a^{\sigma_0-1}) = b \in C.$$

Combining (1) and (2), we see that  $(n-1)[a^{\sigma_0-1}, x] = [b, x] = 0$  for all  $x \in R$ , and thus by (\*\*),  $[a^{\sigma_0-1}, x] = 0$ , for all  $x \in R$ . This, however, contradicts the minimality of  $\sigma_0$  in equation (1), so  $\sigma_0 = 1$ . Therefore, by (1),  $[a, x] = 0$  for all  $x \in R$ , which contradicts  $a \notin C$ . This contradiction proves  $N \subseteq C$ , and the theorem now follows from Theorem 1.

**Theorem 3.** Suppose that  $R$  is a generalized  $n$ -potent ring with all idempotents central. Suppose further that for all  $a \in N, x \in R$ ,

$$(n-1)[a, x] = 0 \quad \text{implies} \quad [a, x] = 0.$$

Then  $R = C$  or  $R = N$ .

*Proof.* By Lemma 2,  $N$  is an ideal of  $R$ . Suppose  $R \not\subseteq C \cup N$ , and let  $x$  be an arbitrary element of  $R \setminus (C \cup N)$ . Then, by (\*),  $x^n - x \in C \cap N$ . Hence, as in the proof of Lemma 1,  $x^m = x^m e$  for some positive integer  $m$  and some idempotent  $e$  of  $R$ . By hypothesis, we have  $e \in C$ . Since  $x \notin N$ ,  $e$  is nonzero. Now, we consider the Peirce decomposition

$$R = Re \oplus A,$$

where  $A = \{a - ae; a \in R\} = \{a \in R; ae = 0\}$ , which will be denoted by  $R(1 - e)$ . Obviously

$$C = Ce \oplus C(1 - e), \quad N = Ne \oplus N(1 - e)$$

and,  $Ce$  (resp.  $Ne$ ) coincides with the set of all central (resp. nilpotent) elements of  $Re$ . Further

$$\begin{aligned} C \cap N &= (Ce \cap Ne) \oplus (C(1 - e) \cap N(1 - e)), \\ R \setminus (C \cup N) &\supseteq Re \setminus (Ce \cup Ne). \end{aligned}$$

From this, an easy computation enables us to see that  $Re$  is a generalized  $n$ -potent ring with the identity  $e$ . Moreover, for all  $a \in N$ ,  $b \in R$ , we have  $ae \in N$ , and hence by hypothesis,

$$(n-1)[ae, be] = 0 \quad \text{implies} \quad [ae, be] = 0.$$

Now by Theorem 2, we see that  $Re$  is a commutative ring, whence  $Re \subseteq C$ . We write  $x = xe + (x - xe)$ . Then  $x^m = x^m e + (x - xe)^m$ . Since  $x^m = x^m e$ , we have  $(x - xe)^m = 0$ . Hence  $x - xe \in N$ .

We set  $a = x - xe$ . Since  $x \notin C$  and  $xe \in C$  (since  $Re \subseteq C$ ), we have  $a \notin C$ . Suppose that

$$a^\sigma \in C \quad \text{for all} \quad \sigma \geq \sigma_0, \quad \sigma_0 \text{ minimal.}$$

Then  $\sigma_0 > 1$ ,  $a^{\sigma_0-1} \notin C$ , and  $e + a^{\sigma_0-1} \notin N \cup C$ . Hence

$$(e + a^{\sigma_0-1})^n - (e + a^{\sigma_0-1}) \in C.$$

Since  $ea^{\sigma_0-1} \in eR \subseteq C$  and  $(a^{\sigma_0-1})^m \in C$  for all  $m \geq 2$ , we have,

$$(e + a^{\sigma_0-1})^n \in C, \quad \text{and so} \quad e + a^{\sigma_0-1} \in C.$$

This implies  $a^{\sigma_0-1} \in C$ , which is a contradiction. Therefore, it follows that  $R = C \cup N$ . Thus, we obtain that  $R = C$  or  $R = N$  (see above Remark (a)).

We are now in a position to prove our main result.

**Main Theorem.** *Suppose that  $R$  is a generalized  $n$ -potent ring with all idempotents central. Suppose further that*

- (i) *for all  $a, b \in N$ ,  $[a, b] = [a, b]^q$  for some integer  $q > 1$ ; and*
- (ii)  *$(n-1)[a, x] = 0$  implies  $[a, x] = 0$ , for all  $a \in N$ ,  $x \in R$ .*

*Then  $R$  is commutative (and conversely).*

*Proof.* Let  $a, b \in N$ . By Lemma 2(iii), the commutator  $[a, b] \in N$ , and thus for some positive integer  $r$ ,  $[a, b]^r = 0$ . Moreover, by hypothesis (i),  $[a, b] = [a, b]^q$  for some integer  $q > 1$ . Hence, as is seen in above Remark, we have  $[a, b] = 0$  and, therefore,  $N$  is commutative. By Theorem 3,  $R = C$  or  $R = N$ , and the theorem thus follows.

Jacobson's Theorem for fixed  $n$ , is an immediate corollary of our main theorem, since in that case  $N = \{0\}$ . So all idempotents  $e$  in  $R$  are central. This follows because  $(ex - exe)^2 = 0 = (xe - exe)^2$ . We state this result as

**Corollary 1.** *Suppose that  $R$  is a ring such that for all  $x \in R$ ,  $x^n = x$ ,  $n > 1$  a fixed integer. Then  $R$  is commutative.*

We also have the following additional corollary:

**Corollary 2.** *Suppose  $R$  is a generalized  $n$ -potent ring with all idempotents central and with commuting nilpotents. Suppose further that the set  $N$  of nilpotents is  $(n - 1)$ -torsion-free. Then  $R$  is commutative.*

*Proof.* By Lemma 2(ii),  $N$  is an ideal of  $R$ , and so  $[a, x] \in N$ , for all  $a \in N$  and  $x \in R$ . Therefore, hypothesis (ii) of the main theorem is satisfied (since  $N$  is  $n - 1$ -torsion-free), and the corollary follows.

We conclude with the following examples which show that our main theorem need not be true if any one of the hypotheses is deleted.

**Example 1.** Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} : 0, 1 \in GF(2) \right\};$$

and let  $n = 2$ .

This example shows that the hypothesis that all of the idempotents are central cannot be deleted from the main theorem.

**Example 2.** Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} : a, b, c \in GF(4) \right\};$$

and let  $n = 2$ .

This example shows that the hypothesis that the ground ring  $R$  is generalized  $n$ -potent cannot be deleted from the main theorem.

**Example 3.** The ring of all *strictly* upper triangular  $3 \times 3$  matrices over  $GF(3)$ , with  $n = 3$ , shows that hypothesis (i) of the main theorem cannot be deleted.

**Example 4.** The ring in Example 2, but now with  $n = 7$ , shows that hypothesis (ii) of the main theorem cannot be deleted.

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